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Thermal Resistance of a Convectively Cooled Plate with Nonuniform Applied Flux

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Two-dimensional steady conduction within a plate of rectangular cross section is considered. One of the surfaces is convectively coupled to a uniform environment temperature while the opposite face is subjected to a nonzero flux distribution over a portion of its boundary. An analytical solution is presented for a general flux distribution and three specific cases are solved. The solution depends upon the Biot modulus, the plate thickness, and the extent and character of the flux distribution. Graphical results are presented for the thermal constriction resistance over the range of parameters of practical interest.

Nomenclature

a	= width of contact in typical cell
b	= width of the typical cell for analysis
Bi	= Biot modulus $\equiv hb/k$
c	= thickness of typical cell
$C_0, C_1, \ldots C_7$	= constants, defined in text
F	= function dependent upon flux distribution
h	= heat-transfer coefficient
k	= thermal conductivity
L	= length of typical cell normal to cross section
n,m	= integers
q	= heat flux
q_0	= constant in heat flux distribution
Q	= total heat flow
R	= thermal resistance $\equiv (\bar{T}_c - T_f)/Q$
R^*	= nondimensional resistance $\equiv RkL$
R_c^*	= nondimensional constriction resistance
T	= temperature
T^*	= nondimensional temperature $\equiv kL(T-T_f)/Q$
$\bar{T}_c$	= average contact area temperature
T_f	= fluid temperature
u	= nondimensional coordinate $\equiv x/a$
x,y	= Cartesian coordinates
α	= nondimensional thickness $\equiv c/b$
ϵ	= nondimensional contact width $\equiv a/b$
ζ	= nondimensional coordinate $\equiv y/b$
λ	= eigenvalue of analytical solution
μ	= parameter in flux distribution
ξ	= nondimensional coordinate $\equiv x/b$
ψ	= function, defined in text

Introduction

In many engineering situations, a thermal conductor convectively cooled on one face may be subject to a nonuniform heat flux over its opposite face. In particular, this nonuniform flux distribution may be prescribed over only a portion of the surface, with the adjacent surface area remaining essentially adiabatic for thermal analysis purposes. This problem is examined in this work for an arbitrary flux distribution and for the case of steady two-dimensional heat transfer.

The analysis presented in this work will find important application to the design and analysis of solar collectors and collector plates in both the design of solar collectors and in the evaluation of experimental facilities utilized to estimate collector losses due to free convection within the collector. The collector plate is frequently constructed of a thermal conductor having evenly spaced coolant tubes secured to its lower surface. A second important application of this work is to the cooling of banks of electronic circuitry as can be found where extensive use is made of integrated circuit (IC) devices. In order to maintain the temperature of the electronic circuitry below its maximum reliable operating temperature, a knowledge of the thermal resistance of the mounting plate is required.

Restricting the analysis to the case of evenly spaced tubes or IC banks in the aforementioned examples, a single typical cell can be extracted for analysis purposes. This typical cell is delineated by the planes of thermal symmetry existing at the center of the flux distribution and at the midpoint between neighboring tubes or IC banks. The thermal influence of the component of interest, tube, IC bank, etc., will be modeled as a flux distribution over the region of contact of the device with its mating surface. The problem geometry is then that shown in Fig. 1 for a typical cell. A uniform heat transfer coefficient h is considered in this work. The portion of the surface on the device contact plane lying outside of the device contact area is assumed to be impervious to heat transfer.

The problem geometry of Fig. 1 has been examined by several investigators,[1-4] each of which considered restrictive thermal boundary conditions. Van Sant[1] considered the case of a uniform flux contact region and an isothermal contact region conducting to a convectively coupled lower surface. For the isothermal contact case a numerical procedure was required and a plot of the results presented. His interests, however, were only in examining the temperature variation over the convectively cooled surface and are of limited utility in the evaluation of the thermal resistance. For the uniform flux case, only the series representation of the maximum temperature variation over this surface was presented.

Schmitz[2] considered the special case of an isothermal contact strip conducting to a second, isothermal, surface. He used separation of variables in his solution to this problem but presented the series representation for only the heat flow

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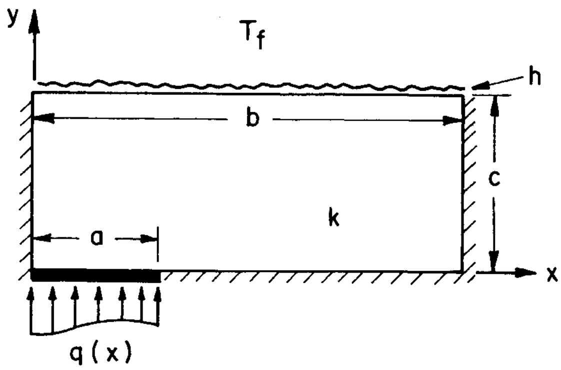

Fig. 1 Typical cell for analysis.

through the conducting member. This same problem was later solved by Costello,[3] who used the theory of conformal transformations to obtain a closed-form solution for this special case. In his solution the conductance for the plate section is determined in terms of complete elliptic integrals of the first kind. The complex computational procedure required to evaluate the pertinent solution parameters, however, makes his results very difficult to use in practice.

Oliveira and Forslund[4] examined the case where both the contact strip and the lower plate surface are convectively coupled through film coefficients to two external fluid temperatures. Again, separation of variables was used and a constriction coefficient was determined. They presented a solution for the uniform flux contact and their results agree with those of Van Sant.[1] The solution where the contact region is convectively coupled to a second fluid temperature, however, produced some unexplained abnormal behavior. The lack of explanation for this somewhat erratic behavior suggests that there are still unresolved questions concerning the convective contact solution.

It is the purpose of this paper to examine the flux prescribed contact case with the heat conducted to a second surface which is convectively coupled to an external fluid. This will be done in a general fashion for arbitrary flux distributions and for any combination of the geometric parameters. Two specific flux distributions in addition to the uniform flux case are examined in detail. One of these provides a very close approximation to the uniform temperature contact situation over a very wide range of the geometric parameters, while the second distribution provides flux concentrations near the contact centerline. The three distributions examined will be useful in estimating limits for the thermal resistance by which most cases of practical interest will be bounded. The thermal resistance is presented for all three cases of practical interest as a fucntion of the geometric parameters and the heat-transfer coefficient h through the Biot modulus Bi.

Problem Solution

Mathematical Statement of the Problem

The geometry to be analyzed is that shown in Fig. 1. A Cartesian coordinate system is set up as shown in the figure with the origin coincident with the center of the contact region. The typical cell half-width is denoted by b with the contact having a half-width of a. The plate thickness is denoted by c. Over the upper surface, the plate communicates thermally with a fluid at a temperature T_f through the convective film coefficient h, while over the contact a flux distribution $q(x)$ symmetric about the vertical axis, is prescribed. The remaining boundaries are assumed to be impervious to heat transfer.

Considering steady-state heat transfer with no internal heat generation, the governing differential equation is Laplace's equation in the two directions considered,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{1}$$

with the boundary conditions given by

$$y=0 \quad 0\leq x\leq a \quad \frac{\partial T}{\partial y} = \frac{-q(x)}{k}$$

$$a\leq x\leq b \quad \frac{\partial T}{\partial y} = 0 \tag{2a}$$

$$y=c \quad 0\leq x\leq b \quad \frac{\partial T}{\partial y} = \frac{-h[T(x,c)-T_f]}{k} \tag{2b}$$

$$x=0 \quad 0\leq y\leq c \quad \frac{\partial T}{\partial x} = 0 \tag{2c}$$

$$x=b \quad 0\leq y\leq c \quad \frac{\partial T}{\partial x} = 0 \tag{2d}$$

In order to maintain generality of the analysis it is useful to nondimensionalize the governing differential equation, Eq. (1), and boundary conditions, Eqs. (2). To effect this normalization, we introduce the following nondimensional variables:

$$\xi \equiv x/b \qquad \zeta \equiv y/b \qquad T^* = kL(T-T_f)/Q \tag{3}$$

where

$$Q = L\int_0^a q(x)\,\mathrm{d}x \tag{4}$$

is the total heat flow rate through the plate over a length L. Using the above definitions the governing differential equation can be written as

$$\frac{\partial^2 T^*}{\partial \xi^2} + \frac{\partial^2 T^*}{\partial \zeta^2} = 0 \tag{5}$$

with boundary conditions.

$$\zeta=0 \quad 0\leq \xi\leq \epsilon \quad \frac{\partial T^*}{\partial \zeta} = \frac{-q(\xi)bL}{Q}$$

$$\epsilon<\xi\leq 1 \quad \frac{\partial T^*}{\partial \zeta} = 0 \tag{6a}$$

$$\zeta=\alpha \quad 0\leq \xi\leq 1 \quad \frac{\partial T^*}{\partial \zeta} = -BiT^*(\xi,\alpha) \tag{6b}$$

$$\xi=0 \quad 0\leq \zeta\leq \alpha \quad \frac{\partial T^*}{\partial \xi} = 0 \tag{6c}$$

$$\xi=1 \quad 0<\zeta<\alpha \quad \frac{\partial T^*}{\partial \xi} = 0 \tag{6d}$$

where the additional parameters have been introduced

$$\epsilon \equiv a/b \qquad \alpha = c/b \tag{7}$$

as well as the Biot modulus defined by

$$Bi \equiv hb/k \tag{8}$$

It can be readily seen now by examination of Eqs. (5) and (6) that the solution for the temperature field will be dependent upon the four nondimensional parameters $q(\xi)bL/Q$, Bi, ϵ, and α, in addition to the two spatial coordinates.

Analytical Solution

In solving the thermal problem described above, solutions are sought to Eq. (5) that satisfy the appropriate boundary conditions, Eqs. (6). Following the classical method of separation of variables,[5] the general solution to Eq. (5), which satisfies conditions Eqs. (6), can be written as

$$T^*=\alpha-\zeta+\frac{1}{Bi}\sum_{n=1}^{\infty}\frac{2}{n\pi}\left[\frac{bL}{Q}\int_0^{\epsilon}q(\xi)\cos(n\pi\xi)\,d\xi\right]$$
$$\times\cos(n\pi\xi)\left[\psi_n\cosh(n\pi\zeta)-\sinh(n\pi\zeta)\right] \quad (9)$$

where ψ_n is defined by

$$\psi_n=\frac{n\pi\cosh(n\pi\alpha)+Bi\sinh(n\pi\alpha)}{n\pi\sinh(n\pi\alpha)+Bi\cosh(n\pi\alpha)} \quad (10)$$

The above temperature distribution is at this point totally general in that arbitrary $q(x)$ [or $q(\xi)$] can be considered in this form of the solution.

Thermal Resistance

In the analysis presented herein the quantity of interest is the thermal resistance from the contact region to the external fluid temperature T_f. Following the usual definition, the overall thermal resistance is given by

$$R\equiv(\bar{T}_c-T_f)/Q \quad (11)$$

where

$$\bar{T}_c=\frac{1}{\epsilon}\int_0^{\epsilon}T(\xi,0)\,d\xi \quad (12)$$

is the average temperature of the contact region. Multiplying Eq. (11) by the product kL yields a dimensionless resistance

$$R^*\equiv RkL=kL(\bar{T}_c-T_f)/Q \quad (13)$$

In view of our definition of T^*, Eq. (3), the dimensionless thermal resistance becomes simply

$$R^*=\frac{1}{\epsilon}\int_0^{\epsilon}T^*(\xi,0)\,d\xi \quad (14)$$

which is the average dimensionless temperature over the contact region. Using the expression developed earlier for the temperature distribution in Eq. (14) yields the result

$$R^*=\alpha+1/Bi$$
$$+\sum_{n=1}^{\infty}\frac{2}{n^2\pi^2}\left[\frac{bL}{Q}\int_0^{\epsilon}q(\xi)\cos(n\pi\xi)\,d\xi\right]\psi_n\frac{\sin(n\pi\epsilon)}{\epsilon} \quad (15)$$

where ψ_n is defined in Eq. (10).

A dimensionless thermal constriction resistance can be obtained by subtracting from the thermal resistance of Eq. (15), the resistance of the slab which results from one-dimensional heat conduction from the surface $\xi=0$ to the fluid at temperature T_f. The one-dimensional resistance is given in nondimensional form by

$$R^*_{1-D}=\alpha+1/Bi \quad (16)$$

so that the nondimensional constriction resistance is given by

$$R_c^*=\sum_{n=1}^{\infty}\frac{2}{n^2\pi^2}\left[\frac{bL}{Q}\int_0^{\epsilon}q(\xi)\cos(n\pi\xi)\,d\xi\right]\psi_n\frac{\sin(n\pi\epsilon)}{\epsilon} \quad (17)$$

where ψ_n is defined in Eq. (10).

Special Cases

The result of prime interest in this work, the thermal constriction resistance of the geometry described in Fig. 1, has been determined and is given by Eq. (17) of the previous section. In their present form these results are general in that arbitrary flux distributions have been considered. In this section three specific flux distributions will be treated as special cases of the above general results. The flux distributions to be considered are given by

$$q(x)=q_0(1-u^2)^{\mu} \qquad \mu=-\tfrac{1}{2},0,+\tfrac{1}{2} \quad (18)$$

where u is defined by $u\equiv x/a$. These flux distributions are illustrated graphically in Fig. 2.

As seen by Eq. (17), the selection of a particular distribution will influence only the term in the square parentheses. For convenience this term will be denoted by $F_n(\epsilon)$:

$$F_n(\epsilon)\equiv\frac{bL}{Q}\int_0^{\epsilon}q(\xi)\cos(n\pi\xi)\,d\xi \quad (19)$$

Writing Eq. (19) in terms of u using Eq. (18) and recalling the definition of Q from Eq. (4), Eq. (19) can be rewritten as

$$F_n(\epsilon)=\int_0^1(1-u^2)^{\mu}\cos(n\pi\epsilon u)\,du\Big/\int_0^1(1-u^2)\,du \quad (20)$$

Considering the three values for the parameter, $\mu=-\tfrac{1}{2}$, 0, $+\tfrac{1}{2}$, the integrals appearing in Eq. (20) can be readily evaluated.[6] The resulting expressions for $F_n(\epsilon)$ are presented in Table 1 for the three cases considered.

The nondimensional thermal constriction resistance can now be compactly written in the form

$$R_c^*=\sum_{n=1}^{\infty}\frac{2}{n^2\pi^2}\cdot F_n(\epsilon)\cdot\psi_n\frac{\sin(n\pi\epsilon)}{\epsilon} \quad (21)$$

where $F_n(\epsilon)$ and ψ_n are obtained from Table 1 and Eq. (10), respectively, for the particular case of interest.

Results

The thermal constriction resistance has been obtained for the range of Biot modulus from $Bi=0.01$ through $Bi=100$.

Table 1 Influence of μ on F_n

μ	$F_n(\epsilon)$
$-\tfrac{1}{2}$	$J_0(n\pi\epsilon)$
0	$\sin(n\pi\epsilon)/n\pi\epsilon$
$+\tfrac{1}{2}$	$2J_1(n\pi\epsilon)/n\pi\epsilon$

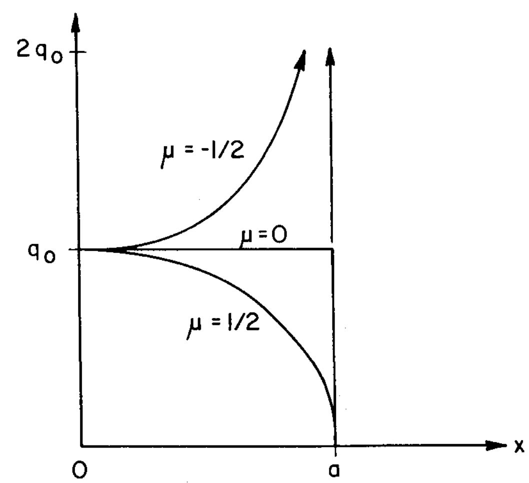

Fig. 2 Flux distributions.

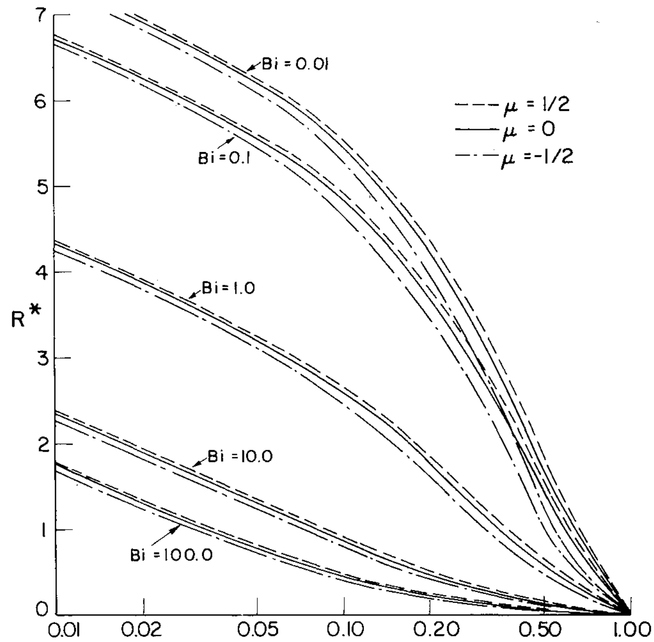

Fig. 3 Thermal constriction resistance for $\alpha = 0.05$.

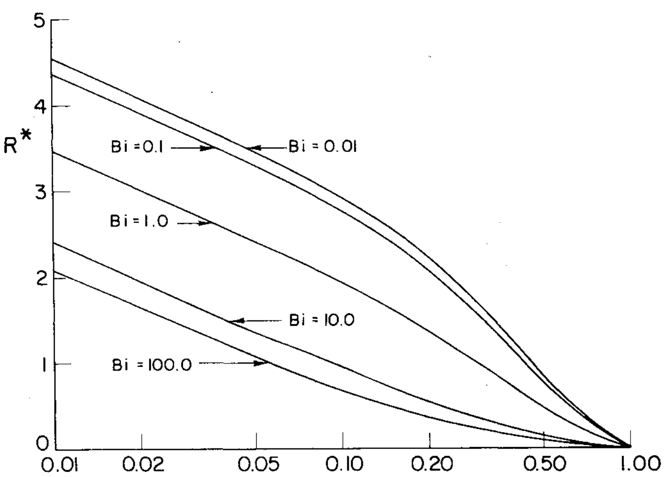

Fig. 4 Thermal constriction resistance for $\alpha = 0.1$.

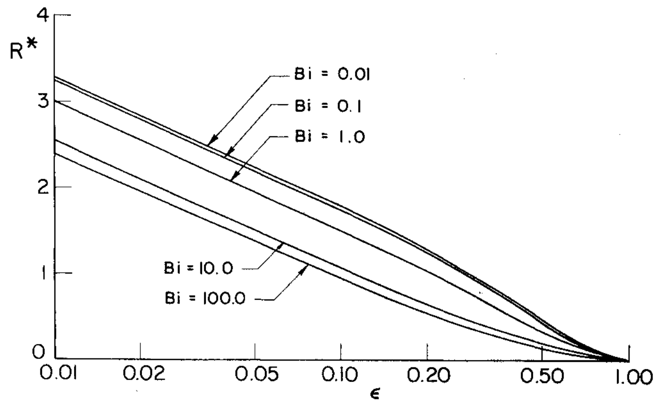

Fig. 5 Thermal constriction resistance for $\alpha = 0.2$.

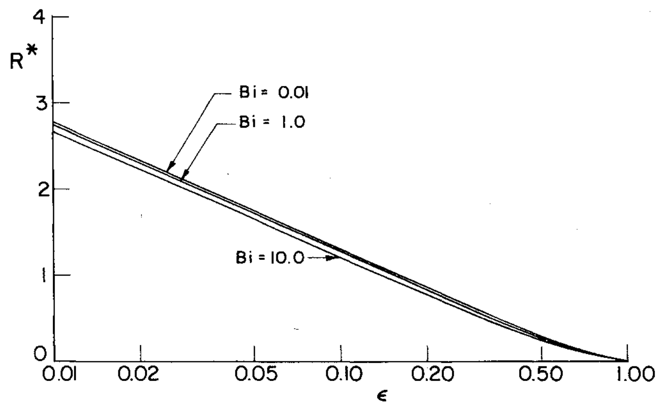

Fig. 6 Thermal constriction resistance for $\alpha = 0.5$.

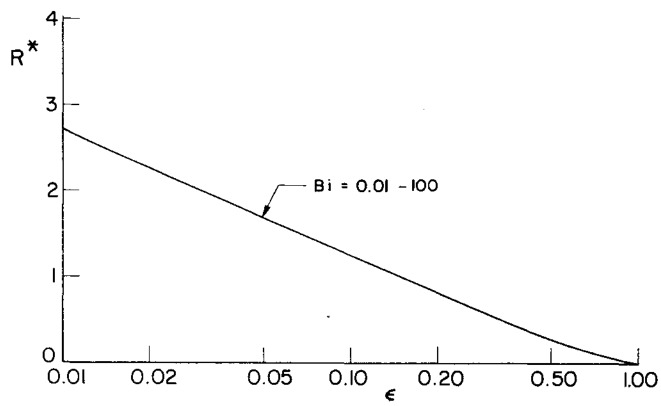

Fig. 7 Thermal constriction resistance for $\alpha = 2.0$.

Dimensionless thickness ratios have been considered over the range $0.05 \le \alpha \le 2.0$, and the dimensionless contact size considered over the range of $0.1 \le \epsilon \le 1.0$. These results are presented in Figs. 3 through 7. These figures correspond to various values of the thickness ratio α with each figure indicating the dependence of R_c^* on the Biot modulus Bi and on the relative contact size ϵ.

It can be observed from these figures that the constriction resistance is largest for the parameter combination of small ϵ, small α, and small Bi. The increase in R_c^* with decreasing values of ϵ is monotone for all cases as is expected. The dependence of R_c^* on α, however, is dependent upon the particular Biot modulus under consideration.

For small values of the Biot modulus, corresponding to poorly conducting solid/fluid interfacial behavior, the heat flow is forced to spread more uniformly over the upper, convective surface. This can only be accomplished by a lateral heat flow in the positive x direction and consequently for small relative thicknesses α there is a large constriction influence. As the thickness increases and the lateral conductive resistance therefore decreases, the thermal constriction resistance also decreases as observed by comparison of the $Bi = 0.01$ curves for example as the thickness ratio is increased from 0.05 to 2.0.

For large values of the Biot modulus, however, the dependence of R_c^* on α is the opposite of that described above. For a very large Biot modulus, for example, the upper, convectively coupled surface remains very nearly uniform in temperature at the fluid temperature. In this case very large fluxes can be tolerated at the convective surface with only a slight temperature rise above T_f. For very thin members, α small, the solid region constitutes a thermal short circuit and the constriction resistance is small. As the thickness of the member increases, the conductive resistance of the solid portion increases and this causes a larger fraction of the total heat flow to pass laterally prior to leaving the region at the convective surface. Therefore, for a large Biot modulus, the thermal constriction resistance increases as the thickness ratio increases.

When the thickness ratio is increased beyond $\alpha \doteq 0.5$, the dependence on the Biot modulus vanishes and the trends established above for large and small Biot modulus converge to a single value. The dependence of the constriction resistance on α also vanishes in this limit and the thermal constriction resistance becomes dependent on the single parameter ϵ, the relative contact size.

The influence of the three different flux distributions shown in Fig. 2 is presented in Fig. 3 for the case where $\alpha = 0.05$. The departures from the uniform flux distribution results are largest for this case and it can be seen from the figure that the influence is small despite the widely differing nature of the flux distributions. The insensitivity to the nature of the flux distribution is attributed to the averaging procedure used in basing the thermal constriction resistance on the average contact temperature. This is a desirable effect since in adding components thermally in series, it is the average temperature which two contacting surfaces will have in common. While the percentage difference in results due to the different flux distributions increases as $\epsilon = 1$ is approached, this is of little consequence since here the one-dimensional resistance is the dominant one. It is interesting to note, however, that the constriction resistance approaches zero for all flux distributions considered as ϵ approaches unity. This result is apparent upon examination of Eq. (17) but is not an obvious one based solely on physical reasoning. This peculiarity is also attributed to the averaging process carried out in arriving at Eq. (17).

Conclusions and Discussion

The thermal constriction resistance corresponding to a partially flux prescribed surface conducting to a second surface which is convectively coupled to a fluid environment temperature has been determined in this work. The problem has been solved in a general fashion for an arbitrary flux distribution over the contact region and three flux distributions have been specifically examined. Solutions have been obtained for the range of the Biot modulus $0.01 \leq Bi \leq 100.0$ and for the thickness range $0.05 \leq \alpha \leq 2.0$. For each of the above variable combinations the relative contact size influence was determined and plotted for the range $0.01 \leq \epsilon \leq 1.0$.

It was found that the thermal constriction resistance, when based on the average contact temperature, is relatively insensitive to the precise nature of the applied flux distribution over the range of primary interest of small ϵ. This insensitivity is attributed to the averaging of the contact temperature.

It was also found that the maximum thermal constriction effect is obtained for the parameter combination where both the Biot modulus and the width ratio are smallest. For small values of the Biot modulus, the influence of increasing the thickness ratio is to decrease the thermal constriction resistance. Conversely the minimum thermal constriction resistance is obtained for situations where the Biot modulus is large and the thickness ratio is small. For large values of the Biot modulus, the influence of increasing the thickness ratio is also to increase the thermal constriction resistance. For thickness ratios larger than 0.5, the above two trends approach their common asymptote and an insensitivity of R_c^* to both α and Bi is exhibited.

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